A Note on Always Decidable Propositional Forms

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Abstract

We ask the following question: If all instantiations of a propositional formula $A(x_1, ..., x_n)$ in n propositional variables are decidable in some sufficiently strong recursive theory, does it follow that A is tautological or contradictory? and answer it in the affirmative. We also consider the following related question: Suppose that for some propositional formula $A(x_1, ..., x_n)$, there is a Turing program P such that $P([\phi_1], ..., [\phi_n]) \downarrow= 1$ iff $\mathbb{N} \models A(\phi_1, ..., \phi_n)$ and otherwise $P([\phi_1], ..., [\phi_n]) \downarrow= 0$ (where $[\phi]$ denotes the Gödel number of ϕ), does it follow that the truth value of $A(\phi_1, ..., \phi_n)$ is independent of $\phi_1, ..., \phi_n$ and hence that A is tautological or contradictory?

1 Decidability in PA and related systems

Definition 1. Let T be a theory. A propositional formula $A(x_1, ..., x_n)$ is always decicable in T iff T decides every sentence of the form $A(\phi_1, ..., \phi_n)$, where $\phi_1, ..., \phi_n$ are closed formulas (without free variables) in the language of T.

We formulate our claims for the case of PA, but they can be transfered to arbitrary recursive axiom systems that allow Gödel coding.

Lemma 2. For each $n \in \omega$, there is a set of n mutually exclusive non-refutable formulas, i.e. a set $\{\theta_1, ..., \theta_n\}$ of closed $\mathcal{L}_{PA}$ formulas such that no $\neg\theta_i$ is provable in PA and such that $\theta_i \rightarrow \bigwedge_{j=1, j\neq i}^{n} \neg\theta_j$ is provable in PA for $i \in \{1, ..., n\}$.

Proof. We write $\phi <_p \psi$ for the statement 'There is a PA-proof of $\neg\phi$ and the smallest Gödel number n of such a proof is smaller than the smallest Gödel

number of a proof of $\neg\psi$, provided there is one', i.e. $\exists x(\text{Bew}(x, [\neg\phi]) \wedge \forall y < x \neg\text{Bew}(y, \neg\psi))$, where $\text{Bew}(a, b)$ denotes 'a is the Gödel number of a proof of the closed formula with Gödel number b'. Consider the following system of statements (we confuse formulas with their Gödel numbers):
(1) $\bigwedge_{i=2}^{n} z_1 <_p z_i$
(2) $\bigwedge_{i=1,i\neq 2}^{n} z_2 <_p z_i$
... (n) $\bigwedge_{i=1,i\neq n}^{n} z_n <_p z_i$
Applying the Gödel fixpoint theorem generalized to n-tuples of formulas (see e.g. [1]), we get statements $\theta_1, ..., \theta_n$ such that

$$(*)\ \theta_i \leftrightarrow \bigwedge_{j=1,j\neq i}^{n} \theta_i <_p \theta_j$$

is provable in PA for each $i \in \{1, 2, .., n\}$. We claim that $\{\theta_1, ..., \theta_n\}$ is as desired.
First, if θ_i and θ_j are both true (where $i \neq j$), then there are by (*) (Gödel numbers of) proofs β_i for θ_i and β_j for θ_j. Now, again by (*), we have $\beta_i < \beta_j$ and $\beta_j < \beta_i$, which is impossible. Hence θ_i implies $\neg\theta_j$ for all $j \neq i$. This argument can easily be carried out in PA.
Second, suppose that $\neg\theta_i$ is provable in PA for some $i \in \{1, ..., n\}$. If $\neg\theta_i$ is provable, then there is $j \in \{1, ..., n\}$ such that $\neg\theta_j$ is provable and the minimal Gödel number of a proof of $\neg\theta_j$ is minimal among the minimal Gödel numbers of proofs of $\neg\theta_k$ for $k \in \{1, ..., n\}$. Let β be the minimal Gödel number of a proof of $\neg\theta_j$. Then PA proves $\neg\theta_j$. Moroever, it is easily provable in PA that no $k' < k$ is a proof for any of the θ_l, $l \in \{1, ..., n\}$. Hence, by (*), PA proves θ_j, so PA proves $\theta_j \wedge \neg\theta_j$, a contradiction.

□

Lemma 3. For each $n \in \omega$, there are n formulas $\phi_1, ..., \phi_n$ in the language of arithmetic such that for no Boolean combination C of any $n-1$ of them, $PA + C$ decides the remaining one.

Proof. Let $n \in \omega$. By Lemma 2, pick a set $S := \{\theta_1, ..., \theta_{2^n}\}$ of 2^n non-refutable, mutually exclusive formulas. We will construct $\phi_1, ..., \phi_n$ as disjunctions $\bigvee R$ over subsets of S. By choice of the θ_i, it is clear that $\theta_i \implies \bigvee R$ iff $\theta_i \in R$: Clearly, if $\theta_i \in R$, then $\theta_i \implies \bigvee R$; on the other hand, θ_i implies $\neg\theta_j$ for all $j \neq i$, so $\theta_i \implies \neg \bigvee R$ if $\theta_i \notin R$.
Let f be some bijection between $\mathfrak{P}(\{1, 2, ..., n\})$ and S. We proceed to define subsets $S_1, ..., S_n$ of S as follows: We put θ_i in S_j iff $j \in f^{-1}(\theta_j)$. Hence each subset of $\{1, 2, ..., n\}$ is 'marked' as the set of j for which S_j contains a particular θ_i. Set $\phi_i := \bigvee S_j$. We claim that $\{\phi_i | 1 \leq i \leq n\}$ is as desired.
To see this, consider a combination $\bigwedge_{i=1}^{n} \delta_i\phi_i$ where each δ_i is either $\neg$ or nothing (i.e. each ϕ_i appears once, either plain or negated). Then $E := \{i | 1 \leq i \leq n \wedge \delta_i \neq \neg\}$ is a subset of $\{1, ..., n\}$. Let $\theta_j = f(E)$.

Then by what we just observed, θ_j implies all elements of E and implies the negation of all elements of $S \setminus E$. Hence θ_j implies $\bigwedge_{i=1}^{n} \delta_i \phi_i$ (and this implication is provable in PA). Now, if $PA + \bigwedge_{i=1}^{n} \delta_i \phi_i$ was inconsistent, so was $PA + \theta_j$. But then, PA would prove $\neg\theta_j$, contradicting the choice of θ_j. Hence $PA + \bigwedge_{i=1}^{n} \delta_i \phi_i$ is consistent. As $\bigwedge_{i=1}^{n} \delta_i \phi_i$ was arbitrary, $\{\phi_1, .., \phi_n\}$ is indeed as desired. □

Remark: This is a generalization of a construction for the case $n = 2$ given in [3] (p. 19), there attributed to E. Jerabek.

Definition 4. A set S of closed $\mathcal{L}_{PA}$-formulas is independent iff for no finite $S' \subseteq S$, $\phi \in S \setminus S'$ and no Boolean combination C of S', $PA + C$ decides ϕ.

Lemma 5. If S is a finite set of closed $\mathcal{L}_{PA}$ formulas, then S is and independent over PA, iff for every Boolean combination C of the elements of S (conjunction in which each element of S appears once, either plain or negated), $PA + C$ is consistent (provided PA is consistent).

Proof. If some combination C was inconsistent and $\psi_1, ..., \psi_{n-1}$ were the first $n-1$ conjuncts of C (i.e. $\phi_1, ..., \phi_{n-1}$, either plain or negated), then ϕ_n would be decided by $\psi_1, ..., \psi_{n-1}$, contradicting the assumption of independence. □

Theorem 6. Every always decidable formula is either tautological or contradictory, i.e.: Let $A(x_1, ..., x_n)$ be a propositional formula in n propositional variables $x_1, ..., x_n$. Assume that for each n-tuple of $\mathcal{L}_{PA}$-formulas without free variables $(\phi_1, ..., \phi_n)$, we have that PA decides $A(\phi_1, ..., \phi_n)$ (i.e. PA either proves the sentence or refutes it). Then A is either a tautology or contradictory.

Proof. Write A in disjunctive normal form. Supppose A is neither tautological nor contradictory. Let $B_1 : \{x_1, ..., x_n\} \to \{0, 1\}$ be an assignment of truth values to the proposition variables that makes A true and $B_2 : \{x_1, ..., x_n\} \to \{0, 1\}$ another one that makes it false. By Lemma 3, let $\{\phi_1, ..., \phi_n\}$ be an independent set of $\mathcal{L}_{PA}$-formulas of cardinality n. Let C_1 and C_2 be the Boolean combinations corresponding to B_1 and B_2, respectively. Then $PA+C_1$ and $PA+C_2$ are both consistent by Lemma 5; however, in $PA+C_1$, $A(\phi_1, ..., \phi_n)$ is true and in $PA+C_2$, $A(\phi_1, ..., \phi_n)$ is false. Hence $A(\phi_1, ..., \phi_n)$ is not decidable in PA, contradicting the assumption. □

2 Algorithmical Decidability of Propositional Forms

We ask a question analogous to that of the preceeding section, where decidability is now taken to mean decidability by a Turing machine: Sup-

pose that for some propositional formula $A(x_1, ..., x_n)$, there is a Turing program P such that $P([\phi_1], ..., [\phi_n]) \downarrow = 1$ iff $\mathbb{N} \models A(\phi_1, ..., \phi_n)$ and otherwise $P([\phi_1], ..., [\phi_n]) \downarrow = 0$, does it follow that the truth value of $A(\phi_1, ..., \phi_n)$ is independent of $\phi_1, ..., \phi_n$ and hence that A is tautological or contradictory? It turns out that the answer is yes:

Theorem 7. Let A be a propositional form and let P be a Turing program such that $P([\phi_1], ..., [\phi_n]) \downarrow = 1$ iff $\mathbb{N} \models A(\phi_1, ..., \phi_n)$ and otherwise $P([\phi_1], ..., [\phi_n]) \downarrow = 0$. Then A is tautological or contradictory.

Proof. Assume that P is such a program for a propositional formula A. We build a recursive extension T of PA that can roughly be stated as $PA +$ 'P is always right'. As 'P is always right' is true by assumption, T is consistent. T consists of PA together with the sentence $S_{(\phi_1,..,\phi_n)} := (A(\phi_1, ..., \phi_n) \to P([\phi_1], ..., [\phi_n]) \downarrow = 1) \wedge (\neg A(\phi_1, ..., \phi_n) \to P([\phi_1], ..., [\phi_n]) \downarrow = 0)$ for every n-tuple $(\phi_1, ..., \phi_n)$ of closed formulas. Clearly, T is recursive.
Now, as, by assumption, P halts with output 0 or 1 on every n-tuple $(\phi_1, ..., \phi_n)$ of closed formulas, PA will prove this for every single instance; moreoever, T will, via the extra assumptions, know that P decides correctly and hence decide $A(\phi_1, ..., \phi_n)$ for every such n-tuple. As Theorem 6 is valid for recursive extensions of PA, it is valid for T, so A is either a tautology or contradictory, as desired. $\square$

References

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